frequencies, τ_{kn}, for which the liquid moment has a local maximum. Fortunately, the magnitude of the maximum decays rapidly with n and only the first three values of n need be considered.

In Ref. 9, it is shown that the corresponding functions in Eq. (3) for inviscid flow in a spheroid are products of associated Legendre functions. For the forced coning motion of a spheroid, all the d_k's are zero except for d_2, which is a real function of τ, ϵ, and f_s, where f_s is the fineness ratio of the spheroid. Thus, the liquid moment is completely determined by d_2 and the function F_2. Once again, the appropriate boundary-layer analysis[9] inserts an imaginary part in d_2 and a nonzero side-moment results. F_2, however, has only one eigenvalue, τ_2, and the side moment has one maximum when τ is near τ_2. Since the liquid moment for a cylinder has a double infinity of local maxima, the situation for a spheroid is much simpler.

Discussion

The liquid side moment produced by liquid in a cylindrical container has significant local maxima at a number of eigenfrequencies. Liquid in a spheroidal container has only one significant local maximum in its liquid side moment and this is at the eigenfrequency τ_2. In Fig. 1, τ_2 is plotted as a function of f_s for f_s between 1 and 4. In flight, τ is always less than 0.4 and τ_2 is greater than this value for most of this range of fineness ratio. Thus, the primary flight stability result of this note is that low-viscosity liquids in spheroidal containers will have no flight instability if f_s is less than 1 or greater than 1.5.

Cooper[16] has coded the complete calculation for spheroid side moment for a VAX mainframe computer. In Fig. 2, C_{LSM} is plotted as a function of frequency for $f_s = 1.5$, $Re = 500{,}000$, and $\epsilon = 0$, 0.01. For zero damping, a maximum side moment of 44 occurs at $\tau_2 = 0.385$. The presence of a small amount of undamping represented by $\epsilon = 0.01$ reduces this maximum value to 19.

We can select the fineness ratio f_c of a cylindrical cavity so that its τ_1 is equal to the τ_2 of a spheroidal cavity with fineness ratio f_s. This process defines the relation between fineness ratios of spheroidal cavities and their "equivalent" cylindrical cavities.

We compare liquid side-moment coefficients in Fig. 3 for a liquid-filled spheroidal cavity with $f_s = 1.5000$ and its equivalent liquid-filled cylindrical cavity, $f_c = 1.6236$. These curves are very similar and we see that the amplitudes of the moments for the two different slopes are nearly the same when the resonance occurs at the same frequency.

Conclusions

1) Liquid moment coefficients have been computed for fully-filled spheroidal cavities as functions of fineness ratio, coning frequency, and Reynolds number.

2) Only one eigenfrequency, τ_2, is important for side-moment calculations, and it can have adverse effects on flight stability for a very limited range of fineness ratios ($1 < f_s < 1.5$).

3) The amplitudes of liquid side-moment coefficients for spheroidal cavities are very similar to those for cylindrical cavities with the same primary resonance frequency.

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Optimum Design of Structures in a Fuzzy Environment

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Introduction

THE theory of fuzzy sets is developed for a domain in which descriptions of activities and observations are "fuzzy," in the sense that there are no well-defined boundaries of the set of activities or observations to which the descriptions apply. The theory enables one to structure and describe activities and observations that differ from each other vaguely, to formulate them in models, and to use these models for various purposes, such as problem-solving and decision-making.

It is well known that, in practice, designers are often forced to state their design problems in precise mathematical terms rather than in terms of the real world, which may often be im-

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precise by nature. The relationships and statements used may be imprecise not due to randomness but because of fuzziness. This may be due to different reasons: designers might not be able to express their objectives precisely because their utility functions are not definable precisely, or phenomena of the design problem might only be described in a fuzzy way. While "This beam carries a load of 1000 lb with a probability of 0.9" is imprecise because of the randomness in the material properties of the beam, the statement "This beam carries a large load" is imprecise because of the fuzzy meaning of "large load." Examples of statements due to imprecise utility functions are: "Our design costs much lesser than our competitor," and "This structure can withstand a substantially larger load than 1000 lb." It is recognized that one's ability to make precise and significant statements concerning a given system diminishes with increasing complexity of the system.[1]

Fuzzy set theory was initiated by Zadeh in 1965.[2] Since then, for some 10 years, the mathematics of the subject was developed by various researchers, but few applications resulted. However, in recent years, the subject has been applied to a wide variety of scientific areas, such as artificial intelligence (AI) and robotics, image processing, speech recognition, biological and medical sciences, decision theory, economics, geography, sociology, psychology, linguistics, and semiotics. The concept of fuzzy sets has been applied to several engineering systems as well. Uragami et al.[3] considered the problem of guiding a robot using fuzzy instructions. A methodology based on fuzzy sets for transportation network planning has been applied to the design of a bus network in a town.[4] A problem of optimal assignment of employees to workplaces, where data and constraints are verbally defined, has been studied by Kacprzyk.[5] Sommer and Pollatschek[6] have applied fuzzy linear programming to an air-pollution regulation problem. Human factors in the failure of structures were also considered wherein the subjectively assessed parameters were used in making modifications of the evaluation of the probability of failure.[7-9]

Fuzzy set theory can be applied in many phases of structural design. For example, any past structural failure can be analyzed and the possible causes of the failure can be expressed by (vague) linguistic statements. By subjectively assessing the degree of confidence in the truth of each linguistic statement and the importance of the statement with respect to the failure, fuzzy set theory can be used to find the "inevitability" of the exprerienced failure. Use of a similar assessment for the present and future systems (structures) will help in deciding whether the project should be delayed while certain matters are investigated or whether to allow events to proceed and risk an accident. This procedure will help when analyzing even complex systems such as space shuttle launching. Similarly, in creating expert systems, the AI researchers try to express the human expert's knowledge in computer code, usually in the form of IF–THEN rules. For example, in the design of structures, the following statement is valid. IF the deflection is slightly more and the cost is satisfactory, THEN increase the cross section of the members. Of course, knowledge of what is meant by "slightly more" and "satisfactory" also has to be represented. The concepts of fuzzy set theory can be used for this purpose.

In conventional optimum design of structures, the stress induced, for example, may be constrained by an upper bound values as $\sigma \leq \sigma^{(u)}$. In structural control, the forcing frequency is expected to be "substantially away" from the natural frequency of the system. All of these problems involve fuzzy information. In the case of stress, if $\sigma^{(u)} = 20{,}000$ psi, it implies that $\sigma = 20{,}000$ psi is allowable but $\sigma = 20{,}001$ psi is unacceptable. However, there is no substantial difference between $\sigma = 20{,}000$ and 20,001 psi. It is more reasonable that there should be transition stages from absolute permission to absolute impermission when the allowable interval of a physical variable is determined, i.e., the ordinary subset should be replaced by a fuzzy subset along the real axis.

In this work, the concept of optimization in a fuzzy environment is presented along with a method of solving structural optimization problems in the presence of fuzzy information.

Optimization in a Fuzzy Environment

The conventional multivariable structural optimization problem can be stated as

$$\text{Find } X \text{ which minimizes } f(X)$$

subject to

$$g_j(X) \leq b_j; \quad j = 1,2,\ldots,m \tag{1}$$

where b_j denotes the upper bound value on the constraint function $g_j(X)$ with $b_j \geq 0$. This problem in a fuzzy environment can be stated as

$$\text{Find } X \text{ which minimizes } \underset{\sim}{f}\,(X)$$

subject to

$$g_j(X) \underset{\sim}{\in} \underset{\sim}{G}_j; \quad j = 1,2,\ldots,m \tag{2}$$

where the ordinary subset G_j denotes the allowable interval for the constraint function g_j, $G_j = [-\infty, b_j]$, and the wave symbols indicate that the operations or variables contain fuzzy information. If d_j represents the permissible variation of $g_j(X)$ about b_j, then $\underset{\sim}{G}_j = [-\infty, b_j + d_j]$. The constraint $g_j \underset{\sim}{\in} \underset{\sim}{G}_j$ means that g_j is a member of a fuzzy subset $\underset{\sim}{G}_j$ in the sense of $\mu_{Gj}(g_j) > 0$. The fuzzy feasible region is defined by considering all the constraints as

$$\underset{\sim}{S} = \bigcap_{j=1}^{m} \underset{\sim}{G}_j \tag{3}$$

and the membership degree of any design vector X to fuzzy feasible region $\underset{\sim}{S}$ is given by

$$\mu_S(X) = \min_{j=1,2,\ldots,m} \{\mu_{G_j}[g_j(X)\} \tag{4}$$

i.e., the minimum degree of satisfaction of the design vector X to all of the constraints.

A design vector X is considered feasible provided $\mu_S(X) > 0$ and the differences in the membership degrees of two design vectors X_1 and X_2 imply nothing but variation in the minimum degree of satisfaction of X_1 and X_2 to the constraints. Thus the optimum solution will be a fuzzy domain D in $\underset{\sim}{S}$ with $\underset{\sim}{f}\,(X)$. The fuzzy domain D is defined by

$$D = \Big\{\mu_f(X)\Big\} \bigcap \Big\{\bigcap_{j=1,2,\ldots,m} \mu_{G_j}[g_j(X)]\Big\} \tag{5}$$

that is,

$$\mu_D(X) = \min\{\mu_f(X), \min_{j=1,2,\ldots,m} \mu_{G_j}[g_j(X)]\} \tag{6}$$

If the membership function of D is unimodal and has a unique maximum, then the optimum solution X^* is one for which the membership function is maximum:

$$\mu_D(X^*) = \max \mu_D(X), \quad X \in D \tag{7}$$

Computational Approach

Let f^* be the optimum value of f for the problem stated in Eq. (1) and $f^* - \Delta f$ the optimum value of f for the problem obtained by replacing b_j by $b_j + d_j$ with $d_j > 0$, $j = 1,2,\ldots,m$ in Eq. (1). It is to be noted that f^* is found with a tighter set of constraints, while $f^* - \Delta f$ is found with a relaxed set of constraints. This is always possible since there will be lower (b_j) and upper (b_j+d_j) limiting values for each response quantity or constraint function $g_j(X)$ in the presence of fuzzy parameters. In this work, for computational convenience, the membership functon of the objective is assumed to vary linearly between f^* and $f^*-\Delta f$, as indicated in Fig. 1a. Thus

$$\mu_f(X)=1, \qquad \text{if } f(X)<f^*-\Delta f$$
$$=1+\left(\frac{f^*-\Delta f-f(X)}{\Delta f}\right), \qquad \text{if } f^*\leq f(X)\leq f^*-\Delta f$$
$$=0, \qquad \text{if } f(X)>f^* \tag{8}$$

Then membership functions for the constraint functions can similarly be represented as (see Fig. 1b).

$$\mu_{g_j}(X)=1, \qquad \text{if } g_j(X)<b_j$$
$$=1-\left(\frac{g_j(X)-b_j}{d_j}\right), \qquad \text{if } b_j\leq g_j(X)\leq b_j+d_j$$
$$=0, \qquad \text{if } g_j(X)>b_j+d_j$$
$$j=1,2,\ldots,m \tag{9}$$

Then the fuzzy optimization problem can be solved using ordinary nonlinear programming techniques as follows:

Find X and λ which maximize λ

subject to

$$\lambda\leq\mu_f(X) \text{ and } \lambda\leq\mu_{g_j}(X); \quad j=1,2,\ldots,m \tag{10}$$

Illustrative Example

The three-bar truss shown in Fig. 2 has been used extensively as an example in structural optimization literature. Let $P=20$, $H=1$, and $\rho=1$. The nonfuzzy optimization problem can be stated as follows:

$$\text{Find } X=\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \text{ which minimizes } f(X)=2\sqrt{2}\,A_1+A_2$$

subject to

$$\sigma_1\leq\sigma^{(u)} \tag{11}$$
$$\sigma_2\leq\sigma^{(u)} \tag{12}$$
$$|\sigma_3|\leq|\sigma^{(\ell)}| \tag{13}$$
$$\delta\leq\delta^{(u)} \tag{14}$$
$$A_i\geq A_i^{(\ell)}; \quad i=1,2 \tag{15}$$

where A_1 and A_2 are the cross-sectional areas of the members,

$$\sigma_1(X)=\text{stress in member } 1=P\left(\frac{A_2+\sqrt{2}A_1}{\sqrt{2}A_1^2+2A_1A_2}\right) \tag{16}$$
$$\sigma_2(X)=\text{stress in member } 2=P\left(\frac{1}{A_1+\sqrt{2}A_2}\right) \tag{17}$$
$$\sigma_3(X)=\text{stress in member } 3=-P\left(\frac{A_2}{\sqrt{2}A_1^2+2A_1A_2}\right) \tag{18}$$

$\delta(X)=$ vertical displacement of the loaded joint

$$=\frac{PH}{E}\left(\frac{1}{A_1+\sqrt{2}A_2}\right) \tag{19}$$

and where E is Young's modulus, H the semiwidth of the truss, and P the load applied.

For the allowable values $\sigma^{(u)}=20$, $\sigma^{(\ell)}=-15$, $\delta^{(u)}=10/E$, and $A_i^{(\ell)}=0.1$ $(i=1,2)$, the optimum solution is given by

$$A_1^*=2/3,\ A_2^*=2\sqrt{2}/3, \text{ and } f^*=2\sqrt{2}=2.8284$$

with the stress [Eq. (11)] and displacement [Eq. (14)] being critical at the optimum point. If the boundaries of the allowable intervals of the response parameters are moved by

$$d_{\sigma^{(u)}}=4,\quad d_{\sigma^{(\ell)}}=3,\quad d_{\delta^{(u)}}=2/E,\quad d_{A_i^{(\ell)}}=0.02;\quad i=1,2$$

the optimum value of the objective functon has been found to be $5\sqrt{2}/3$ (and, hence, $\Delta f=\sqrt{2}/3$) with $A_1^*=5/9$ and $A_2^*=\sqrt{2}\,5/9$.

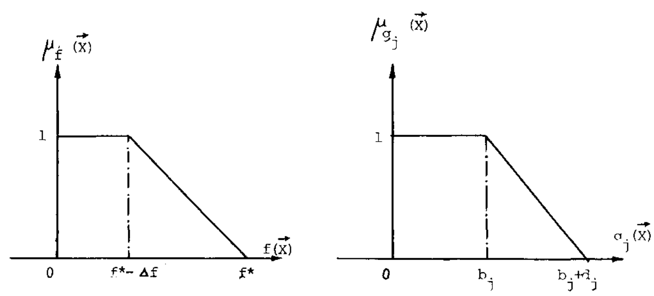
For the fuzzy optimization problem, the membership functions are assumed to have inclined line boundaries. Thus the fuzzy optimization problem can be stated as

$$\text{Find } X=\begin{Bmatrix} \lambda \\ A_1 \\ A_2 \end{Bmatrix} \text{ which maximizes } f(X)=\lambda$$

subject to

$$\lambda\leq 1-\left\{\frac{3(2\sqrt{2}A_1+A_2)-5\sqrt{2}}{\sqrt{2}}\right\} \tag{20}$$

$$\lambda\leq 1-\left\{\frac{(A_2+\sqrt{2}A_1)-(\sqrt{2}A_1^2+2A_1A_2)}{0.2(\sqrt{2}A_1^2+2A_1A_2)}\right\} \tag{21}$$



a) b)

Fig. 1 Membership functions of f and g_j.

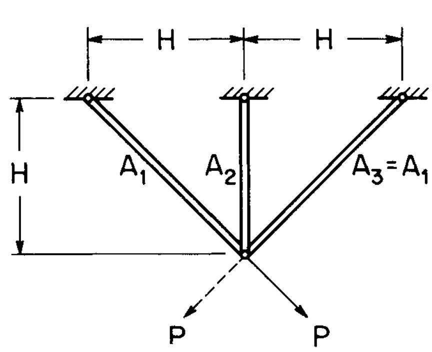


Fig. 2 Three-bar truss.

$$\lambda \le 1 - \left\{ \frac{1-(A_1+\sqrt{2}A_2)}{0.2(A_1+\sqrt{2}A_2)} \right\} \tag{22}$$

$$\lambda \le 1 - \left\{ \frac{4A_2 - 3(2A_1A_2+\sqrt{2}A_1^2)}{0.6(2A_1A_2+\sqrt{2}A_1^2)} \right\} \tag{23}$$

$$\lambda \le 1 - \left\{ \frac{2-(A_1+\sqrt{2}A_2)}{0.2(A_1+\sqrt{2}A_2)} \right\} \tag{24}$$

$$\lambda \le 1 - \left\{ \frac{0.10-A_i}{0.02} \right\}; \quad i=1,2 \tag{25}$$

The solution of this problem is given by $\lambda^* = 0.5226$, $A_1^* = 0.60856$, $A_2^* = 0.86065$ with $f^* = 2.58192$. The constraints of Eqs. (20), (21), and (24) are active at the optimum solution. Notice that the fuzzy formulation has increased the number of design variables by one.

The fuzzy optimum solution has the following interpretation. When the range of the permissible stress in member 1 (σ_1) is stated as 20–24, it implies that a values of $\sigma_1 = 20$ has the maximum degree of satisfaction ($\mu = 1$) and a value of $\sigma_1 = 24$ has a minimum degree of satisfaction ($\mu = 0$). Since the membership function of σ_1 is assumed to have inclined line boundaries, the degree of satisfaction increases linearly from 0 to 1 as σ_1 decreases from 24 to 20. The permissible ranges on all other response parameters, including the objective function, have similar meanings. The fuzzy optimum solution indicates that the maximum level of satisfaction (degree of membership) that can be achieved in the presence of the stated fuzziness in the objective function and constraints is 0.5226.

Conclusion

The description and the optimization of structures containing fuzzy information has been presented. A method of solving fuzzy optimization problems using ordinary nonlinear programming techniques was presented along with a numerical example. The procedures outlined are expected to be useful in situations where doubt arises concerning exactness of concepts, correctness of statements and judgments, degrees of credibility, etc., which are essential for the definition of a crisp design optimization problem. The possible application of fuzzy set theory in the analysis of structural failures and the development of expert systems for structural design are also indicated.

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Iterative Study for Three-Dimensional Finite-Element Stress Analysis

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Introduction

The use of three-dimensional finite-element models based on three-dimensional elasticity theory for the analysis of layered laminates is increasing,[1] but this often results in a large numerical model. The aim of this work is to reduce the total number of degrees of freedom by selecting three-dimensional elements in a coarse mesh.

The Loubignac method[2,3] improves the accuracy of both stresses and displacements in finite-element stress analysis. It produces a stress field that is continuous across interelement boundaries by using average nodal stresses. However, this method cannot be used for problems such as multilayered plates[2] which consist of bonded dissimilar layers because the in-plane stresses are not continuous across the interface of dissimilar materials. A modified algorithm is presented and applied in layered nonhomogeneous structures.

Iterative Formulation

The standard finite-element method in structural mechanics gives a set of equations

$$[K]\{\delta\}_1 = \{R\} \tag{1}$$

where $[K]$ is the structural stiffness matrix, $\{R\}$ is the vector of nodal forces, and $\{\delta\}_1$ is the nodal displacement vector. By using Eq. (1), we can determine the stresses $\{\sigma\}$ in each element by means of the relations

$$\{\sigma\} = [D][B]\{u\}_1 \tag{2}$$

where $[B]$ is the strain displacement matrix, $\{u\}_1$ is a vector of the element nodal displacements which is a component of $\{\delta\}_1$, and $[D]$ is the elastic constitutive matrix. Because of the displacement-based finite-element formulation, the stresses obtained are discontinuous across interelement boundaries: this discontinuity will decrease with mesh refinement. Loubignac et al.[2,3] computed the average stresses $\{\bar{\sigma}_N\}$ at a common node across interelement boundaries. Within an element, let the stresses $\{\bar{\sigma}\}$ be interpolated from the average nodal stresses $\{\bar{\sigma}_N\}$ in the same way that displacements are interpolated from nodal degrees of freedom (DOF) by the shape function $[N]$. Thus,

$$\{\bar{\sigma}\} = [N](\bar{\sigma}_N) \tag{3}$$

When applying the Loubignac iteration in a structure, the determination of the nodal stresses across dissimilar materials by averaging must be avoided. In order to solve more general structural problems, the Loubignac method should be modified. If the orthogonal curvilinear coordinates are r, s, and t, then the stresses and strains in a solid are

$$\{\sigma_r, \sigma_s, \sigma_t, \tau_{st}, \tau_{tr}, \tau_{rs}\}$$

and

$$\{\epsilon_r, \epsilon_s, \epsilon_t, \gamma_{st}, \gamma_{tr}, \gamma_{rs}\}$$

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